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INFINITESIMALLY PI RADICAL ALGEBRAS

BY

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ABSTRACT

The Golod–Shafarevich examples show that not every finitely generated nil algebra A is nilpotent. On the other hand, Kaplansky proved that every finitely generated nil PI-algebra is indeed nilpotent. We generalise Kaplansky's result to include those algebras that are only infinitesimally PI. An associative algebra A is infinitesimally PI whenever the Lie subalgebra generated by the first homogeneous component of its graded algebra $\mathrm{gr}(A) = \bigoplus_{i\geq 1} A^i/A^{i+1}$ is a PI-algebra. We apply our results to a problem of Kaplansky's concerning modular group algebras with radical augmentation ideal.

1. Introduction and statement of results

Throughout this paper, our algebras are based over an arbitrary but fixed field F. One of the famous problems in ring theory is the Kurosh–Levitzki problem: If A is a finitely generated nil algebra, then is A nilpotent? This is false in general, as demonstrated by the examples of Golod and Shafarevich ([G], [GS]). Consequently, it makes sense to address the problem to more specific classes of nil algebras. Modern PI-theory came into existence when Kaplansky ([K1]) gave his affirmative solution to the Kurosh–Levitzki problem in the case when A satisfies a polynomial identity.

The primary aim of this note is first to generalise Kaplansky's theorem, and second, to apply those results to a variant of the Kurosh–Levitzki problem for

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modular group algebras, which was first posed by Kaplansky. The analogous problem for restricted enveloping algebras will be discussed as well. Our prinicipal tool is a deep theorem of Zelmanov ([Z1] and [Z2]), which can be viewed as a Lie-theoretic analogue of Kaplansky's theorem: If L is a finitely generated Lie PI-algebra such that every commutator (of length one or more) in a minimal set of generators is ad-nilpotent, then L is nilpotent. In fact, his result in characteristic zero can be stated more generally (see Theorem 4.4 below).

Recall now that every associative algebra can be viewed as a Lie algebra via the Lie multiplication

$$[a,b]=ab-ba.$$

Our first step will be to prove:

THEOREM A: *Let A be an associative algebra generated by a finite subset X. Suppose that the Lie subalgebra L of A generated by X satisfies the properties:*

1. *every Lie commutator in X is nilpotent in A, and*
2. *L is a PI-algebra.*

Then A is nilpotent.

Lie algebras arising as Lie subalgebras of associative PI-algebras are called special. It is known that every special Lie algebra satisfies a polynomial identity as a Lie algebra (see Corollary to Theorem 6.3.3.11 in [B]). Kaplansky's theorem, therefore, is an immediate consequence of Theorem A.

In fact, Zelmanov's theorem is also a consequence of Theorem A, making the two results more or less equivalent. Indeed, let L be a finitely generated Lie PI-algebra such that every commutator in a minimial set X of generators is ad-nilpotent. Then the associative algebra $\mathrm{Ad}(L)$ of endomorphisms of L generated by the ad-maps $\mathrm{ad}\, x$, x in X, is easily seen to satisfy the hypotheses in Theorem A. Thus $\mathrm{Ad}(L)$ is associatively nilpotent, which is tantamount to the nilpotence of L, as claimed.

Next, we extend Zelmanov's notion of infinitesimally PI groups (see Section 2) to associative algebras.

The associated graded algebra of an associative algebra A is defined by

$$\mathrm{gr}(A)=\bigoplus_{i\geq 1} A^i/A^{i+1}=\bigoplus_{i\geq 1} A_i,$$

with multiplication induced from A in the natural way. Observe that if A is a PI-algebra then certainly so is $\mathrm{gr}(A)$, but not conversely: the assumption that $\mathrm{gr}(A)$ satisfies a (homogeneous multilinear) identity $f(x_1,\ldots,x_m)=0$ merely tells us that

$$f(A^{n_1},\dots,A^{n_m}) \subseteq A^{n_1+\cdots+n_m+1},$$

for all $n_1,\dots,n_m \geq 1$.

Clearly $\mathrm{gr}(A)$ is generated as an associative algebra by its first homogeneous component, A_1. We record now the fact that if A is finitely generated as an associative algebra, then certainly A is finitely generated as an A-module, and hence A_1 is of finite dimension.

Let $\mathcal{L}(A)$ denote the Lie subalgebra of $\mathrm{gr}(A)$ generated by A_1. We shall call A infinitesimally PI if the Lie algebra $\mathcal{L}(A)$ satisfies a polynomial identity. Consequently, we have the following implications:

$$A \text{ is PI } \Rightarrow \mathrm{gr}(A) \text{ is PI } \Rightarrow A \text{ is infinitesimally PI.}$$

Each implication is generally strict (see Section 2 for examples).

Before we state our main result, recall that an associative algebra is said to be radical if it coincides with its Jacobson radical. In particular, nil algebras are radical.

THEOREM B: *Let A be a radical algebra. Suppose that:*

1. *A is finitely generated as an A-module,*
2. *every Lie commutator in a basis X of A_1 is nilpotent in* $\mathrm{gr}(A)$, *and*
3. *A is infinitesimally PI.*

Then A is nilpotent.

COROLLARY: *Let A be a finitely generated nil algebra. If A is infinitesimally PI then A is nilpotent.*

In order to deduce the corollary from Theorem B, it remains to observe that because A is nil, every homogeneous element in $\mathrm{gr}(A)$ is nilpotent — even though it is not clear, *a priori*, that $\mathrm{gr}(A)$ is nil. We stress this point by posing what the author believes to be an open problem:

PROBLEM 1: *If A is a finitely generated nil algebra, does it follow that* $\mathrm{gr}(A)$ *is a nil (or even radical) algebra?*

Because the nilpotence of A clearly forces the nilpotence of $\mathrm{gr}(A)$, any counterexample to Problem 1 would be a non-nilpotent finitely generated nil algebra. The Golod–Shafarevich examples seem to be of no immediate assistance here, however, since they are themselves graded algebras. We also remark that Problem 1 seems to be related to Köthe's Conjecture.

Before proving Theorems A and B, we make two key applications.

2. Kaplansky's problem

Let $\mathcal{A}F[G]$ denote the augmentation ideal of the group algebra $F[G]$ of a group G over a field F of characteristic $p > 0$. The Kurosh–Levitzki problem for group algebras is still open, namely: If G is a finitely generated group such that $\mathcal{A}F[G]$ is nil, does it follow that G is a finite p-group? Recall that G is a finite p-group precisely when $\mathcal{A}F[G]$ is nilpotent. A more general problem, first posed by Kaplansky ([K2]) and recently reiterated by Passman ([P2]), asks:

PROBLEM 2: *If G is finitely generated and $\mathcal{A}F[G] = \mathcal{J}F[G]$, does it follow that G is a finite p-group?*

Here $\mathcal{J}F[G]$ represents the Jacobson radical of $F[G]$. Because G is known to be a p-group (see Lemma 10.1.13 in [P1], for example), the real question here is: is G finite? Using Zelmanov's positive solution of the restricted Burnside problem, it follows from an argument due to Lichtman (see [P1]) that Problem 2 has an affirmative solution for all finitely generated groups of finite exponent. The purpose of this section is to use Theorem B to extend this result to a broader class of groups.

For now, let G be an arbitrary group, and put $A = \mathcal{A}F[G]$. Then, by a slight abuse of our previous notation, $\mathrm{gr}(A)$ is the augmentation ideal of the graded group algebra:

$$\mathrm{gr}(F[G]) = \bigoplus_{i\geq 0} A^i/A^{i+1},$$

where $A^0 = F[G]$. The ith dimension subgroup of G is defined by $D_i(G) = G \cap (1 + A^i)$ for each $i \geq 1$. It is well known that

$$\mathrm{gr}_p(G) = \bigoplus_{i\geq 1} D_i(G)/D_{i+1}(G)$$

forms a restricted Lie algebra over the field $\mathbb{F}_p$ of p-elements with Lie multiplication and p-map induced, respectively, by the operations of commutation and exponentiation by p in G. Furthermore, according to a theorem of Quillen ([Q]), we have

$$\mathrm{gr}(\mathbb{F}_p[G]) \cong u(\mathrm{gr}_p(G)),$$

where $u(\mathrm{gr}_p(G))$ is the restricted (universal) enveloping of algebra of $\mathrm{gr}_p(G)$. Under this isomorphism, we may identify A_1 with the elementary abelian p-group

$G/D_2(G) = D_1(G)/D_2(G)$, and subsequently identify $\mathcal{L}(A)$ with the (ordinary) Lie subalgebra $\mathrm{gr}(G)$ in $\mathrm{gr}_p(G)$ generated by $G/D_2(G)$.

To extend to general F-algebras in the paragraph above, merely replace $\mathrm{gr}_p(G)$ with $F \otimes_{\mathbb{F}_p} \mathrm{gr}_p(G)$.

Zelmanov ([Z2]) calls a residually finite p-group G infinitesimally PI if the Lie algebra $\mathrm{gr}(G)$ is PI. For our present purposes, however, it is neither convenient nor necessary to assume G is residually a finite p-group. Thus, in our notation, G is infinitesimally PI precisely when $\mathcal{A}F[G]$ is infinitesimally PI. Note that here the choice of base field F is irrelevant beyond its characteristic.

Our best attempt at a positive solution of Kaplansky's problem is as follows:

THEOREM C: *Let G be a finitely generated group and let F be a field of characteristic $p > 0$ such that $\mathcal{A}F[G] = \mathcal{J}F[G]$. If G is infinitesimally PI then G is a finite p-group.*

Proof: Because the Jacobson radical of $F[G]$ and the Jacobson radical of its augmentation ideal A always coincide, A is a radical algebra by hypothesis. It remains for us to check that Theorem B applies to the algebra A.

First notice that the finite generation of G together with the identity

$$1 - xy = (1 - x) + (1 - y) - (1 - x)(1 - y)$$

imply that $A = \sum_{g\in G} F(1 - g)$ is finitely generated as an associative algebra. Hence, under our identifications and assumptions, A_1 is a vector space over F with finite basis $G/D_2(G)$. Since G is a p-group, it follows that every homogeneous element in $\mathrm{gr}(G)$ is nilpotent in $\mathrm{gr}(A)$. So, in particular, every commutator in the generators $xD_2(G)$, x in G, is nilpotent in $\mathrm{gr}(A)$. Lastly, as indicated above, A is infinitesimally PI because G is infinitesimally PI. ∎

To illustrate the generality of Theorem C, we now identify some specific classes of infinitesimally PI groups. Let $\Gamma = G_{\hat{p}}$ denote a pro-p completion of G. If G is finitely generated, or more generally if $G/D_2(G)$ is finite, then Γ is unique up to homeomorphism by a theorem of Serre (see [DDMS]). Thus we lose no generality by completing G with respect to its dimension series. It follows that $\mathrm{gr}(G) \cong \mathrm{gr}(\Gamma)$, and so $\mathrm{gr}(F[G]) \cong \mathrm{gr}(F[\Gamma])$ by Quillen's theorem. In particular, G is infinitesimally PI precisely when Γ is infinitesimally PI.

Now, let $F(d)$ denote the free group on d-generators. Then $F(d)_{\hat{p}}$ is the free pro-p group of rank d. A nontrivial element $w \in F(d)_{\hat{p}}$ is called a pro-p identity for a pro-p group Γ if every continuous homomorphism $F(d)_{\hat{p}} \to \Gamma$ maps w to 1. Zelmanov proved in [Z2] that if Γ satisfies a pro-p identity then Γ is infinitesimally

PI. Because $F(d)$ is residually a finite p-group, every (abstract group) law is a special case of a pro-p identity. Observe, as well, that any law satisfied by G is also satisfied by its pro-p completion. Hence, any abstract group G satisfying a law is infinitesimally PI.

In fact, more can be said along this vein. Let H be a subgroup of G and let $a_1, \ldots, a_n$ be elements of G. If $w = w(x_1, \ldots, x_n)$ is a nontrivial element of $F(n)$ then the law $w = 1$ is said to be satisfied on the cosets $a_1H, \ldots, a_nH$ if $w(a_1h_1, \ldots, a_nh_n) = 1$ for all $h_1, \ldots, h_n \in H$. Wilson and Zelmanov ([WZ]) proved: If a group G has a subgroup H of finite index and elements $a_1, \ldots, a_n$ such that a law $w = 1$ holds on the cosets $a_1H, \ldots, a_nH$ then G is infinitesimally PI. It follows from the Baire category theorem that if a pro-p group Γ does not contain a free abstract group of rank 2 then Γ is infinitesimally PI.

COROLLARY: *Let G be a finitely generated group and let F be a field of characteristic $p > 0$ with the property that $\mathcal{A}F[G] = \mathcal{J}F[G]$. Then G is a finite p-group provided any one of the following conditions holds:*

1. *G or $G_{\hat{p}}$ is infinitesimally PI.*
2. *G or $G_{\hat{p}}$ has a subgroup H of finite index and elements $a_1, \ldots, a_n$ such that a law $w = 1$ holds on the cosets $a_1H, \ldots, a_nH$.*
3. *$G_{\hat{p}}$ does not contain a free abstract subgroup of rank 2.*
4. *$G_{\hat{p}}$ satisfies a pro-p identity.*
5. *$G_{\hat{p}}$ is p-adic analytic.*
6. *G satisfies a law.*

To prove the corollary, it remains to remark that p-adic analytic pro-p groups are known to be infinitesimally PI. In fact, Lazard ([L]) proved a pro-p group Γ is p-adic analytic if and only if $\mathrm{gr}_p(\Gamma)$ is finitely generated and nilpotent. Interestingly, it is unknown whether or not such a group satisfies a pro-p identity ([Z2]). Zubkov proved in [Zu], however, that for every odd prime p a free nonabelian pro-p group has no faithful representation by second-order matrices over an associative and commutative profinite ring Λ with unity; in other words, every pro-p subgroup of $\mathrm{GL}_2(\Lambda)$ satisfies some pro-p identity.

In order to give a more general solution to Kaplansky's problem, it may be helpful to note that several other characterisations of p-adic analyticity (in terms of the structures of Γ, $\mathrm{gr}_p(\Gamma)$ and $\mathrm{gr}(F[\Gamma])$) also appear in the literature. See [S2] as a good starting point. In particular, it is known that a finitely generated pro-p group Γ is p-adic analytic if and only if $\mathrm{gr}(F[\Gamma])$ is a PI-algebra (see [R1] or [S1]).

Condition (1) in the corollary truly is the weakest of those listed. Indeed, consider the Nottingham group, Nott(p). Nott(p) is the group of those automorphisms of the algebra of formal power series $\mathbb{F}_p[[t]]$ which act trivially on the factor $t\mathbb{F}_p[[t]]/t^2\mathbb{F}_p[[t]]$. It is known that

$$\mathrm{gr}(\mathrm{Nott}(p)) \cong t\mathbb{F}_p[t] \otimes W_1,$$

where W_1 is the Witt algebra with basis $e_0, \ldots, e_{p-1}$ over $\mathbb{F}_p$ and relations $[e_i, e_j] = (i-j)e_k$ where $k = i+j$ modulo p. Thus Nott(p) is certainly infinitesimally PI. It does not satisfy any pro-p identity, however, since according to [C] it contains all countably based pro-p groups as subgroups, including those which are free. This also implies that the property of being infinitesimally PI is not inherited by subgroups or subalgebras.

Clearly Nott(p) is not p-adic analytic, for otherwise gr(Nott(p)) would be nilpotent; in other words, $\mathrm{gr}(\mathbb{F}_p[\Gamma])$ is not a PI-algebra. Hence

$$A \text{ is infinitesimally PI } \not\Rightarrow \mathrm{gr}(A) \text{ is PI}.$$

Because a group algebra $F[G]$ is a PI-algebra only when G is virtually p-abelian (see [P1]), certainly we also have

$$\mathrm{gr}(A) \text{ is PI } \not\Rightarrow A \text{ is PI}.$$

3. Restricted enveloping algebras

Kaplansky's problem makes sense for restricted enveloping algebras:

PROBLEM 3: *If L is a finitely generated restricted Lie algebra over a field of characteristic $p > 0$ such that $\mathcal{A}u(L) = \mathcal{J}u(L)$, does it follow that L is finite dimensional with nilpotent p-map?*

Definition: Let $\mathcal{F}_p$ denote the class of all finite dimensional restricted Lie algebras with nilpotent p-map.

By a result of Jacobson (see Lemma 2.4 in [RS1] for a proof), the augmentation ideal $\mathcal{A}u(L) = Lu(L)$ of $u(L)$ is nilpotent if and only if L lies in the class $\mathcal{F}_p$. Thus, Problem 3 can be viewed as a more general form of the Kurosh–Levitzki problem for restricted enveloping algebras.

Besides being of interest in its own right, the case of restricted enveloping algebras could lead to a better understanding of group algebras. Indeed, suppose that G is a finitely generated group such that $\mathcal{A}F[G]$ is nil. On the one hand, if Problem 1 fails for $A = \mathcal{A}F[G]$ (which is to say, $\mathcal{A}u(\mathrm{gr}_p(G))$ is not necessarily nil) then the Kurosh–Levitzki problem would fail for group algebras; while on the other hand, if Problem 1 has a positive solution for $A = \mathcal{A}F[G]$, then a positive solution of the Kurosh–Levitzki problem for group algebras would follow from a positive solution of the Kurosh–Levitzki problem for restricted enveloping algebras.

Let L be an arbitrary restricted Lie algebra over a field F of characteristic $p > 0$, and set $A = \mathcal{A}u(L)$. The dimension subalgebras of a restricted Lie algebra are defined by $D_i(L) = L \cap A^i$ for each $i \geq 1$. Dimension subalgebras are studied in detail in [RS2]. The graded restricted Lie F-algebra associated to L is defined by

$$\mathrm{gr}_p(L) = \bigoplus_{i \geq 1} D_i(L)/D_{i+1}(L),$$

with operations induced in the obvious way by those in L. Let $\mathrm{gr}(L)$ denote the (ordinary) Lie subalgebra of $\mathrm{gr}_p(L)$ generated by the first component, $D_1(L)/D_2(L)$. We shall say that L is infinitesimally PI when $\mathrm{gr}(L)$ is a PI-algebra.

Using results found in [RS2], and proceeding as in the proof of the ordinary enveloping algebra case given in [R2], it is not difficult to check the following analogue of Quillen's theorem:

$$u(\mathrm{gr}_p(L)) \cong \mathrm{gr}(u(L)) = \bigoplus_{i \geq 0} A^i/A^{i+1}.$$

Furthermore, we may identify $\mathrm{gr}(L)$ with the Lie subalgebra of $\mathrm{gr}(u(L))$ generated by A^1/A^2. It follows that L is infinitesimally PI precisely when A is infinitesimally PI.

THEOREM D: *Let L be a finitely generated restricted Lie algebra over a field F of characteristic $p > 0$ such that $\mathcal{A}u(L) = \mathcal{J}u(L)$. If L is infinitesimally PI then L lies in the class $\mathcal{F}_p$.*

We remark now that if L is PI then it is certainly infinitesimally PI.

In order to deduce Theorem D from Theorem B, we require some additional information.

LEMMA 3.1: *If H is a p-subalgebra of L then $u(H) \cap \mathcal{J}u(L) \subseteq \mathcal{J}u(H)$.*

Proof: Let H be a p-subalgebra of L. Extend a well-ordered basis of H to a well-ordered basis $\mathcal{B}$ of L in such a way that if $h, x \in \mathcal{B}$, where $h \in H$ and $x \notin H$, then $h \leq x$. Form the Poincaré–Birkhoff–Witt basis of $u(L)$ with respect to $\mathcal{B}$

(see [J] or [SF]), and let M be the kernel of the projection map $u(L) \to u(H)$ induced by mapping each of the PBW monomials involving basis elements not in H to zero. Then $u(L) = u(H) \oplus M$ as left $u(H)$-modules. The final implication is a known general ring-theoretic fact: see Lemma 7.1.3 in [P1] for details. ∎

LEMMA 3.2: *If $\mathcal{A}u(L) = \mathcal{J}u(L)$ then L consists of p-nilpotent elements.*

Proof: Let $x \in L$ and set H to be the p-subalgebra generated by x. Then from the previous lemma, we have

$$\mathcal{A}u(H) \subseteq u(H) \cap \mathcal{A}u(L) \subseteq \mathcal{J}u(H) \subseteq \mathcal{A}u(H),$$

so that $\mathcal{A}u(H) = \mathcal{J}u(H)$. Therefore, because $\mathcal{A}u(H)$ is generated as an associative algebra by x, the quasi-inverse of x can be expressed as a polynomial in x. It follows immediately that x is an algebraic element lying in $\mathcal{J}u(H)$, and hence nilpotent. ∎

The proof of Theorem D now proceeds as in the proof of Theorem C.

4. Proof of Theorems A and B

We shall say that an element x in a Lie algebra is ad-algebraic if the adjoint representation $\operatorname{ad} x$ is algebraic.

LEMMA 4.1: *Any algebraic element in an associative algebra is ad-algebraic. Similarly, any nilpotent element is ad-nilpotent.*

Proof: Suppose first that a is an algebraic element in an associative algebra A over a field F. Let λ_a and ρ_a denote left and right multiplication by a in A. Clearly λ_a and ρ_a satisfy the same polynomial over F that a does; hence, each is an algebraic F-linear endomorphism of A. Thus, since λ_a and ρ_a commute, the subalgebra S of $\operatorname{End}_F(A)$ generated by λ_a and ρ_a is finite-dimensional over F. But $\operatorname{ad} a = \lambda_a - \rho_a \in S$, so that $\operatorname{ad} a$ is also algebraic over F.

Finally, suppose that $a^n = 0$. The binomial theorem yields $(\operatorname{ad} a)^{2n-1} = (\lambda_a - \rho_a)^{2n-1} = 0$. ∎

We shall indicate the nth term of the lower central series of a Lie algebra L by $\gamma_n(L)$. For each x in L, we define $\nu(x)$ to be the largest subscript n such that $x \in \gamma_n(L)$ if n exists, and to be ∞ if it does not.

Let $U(L)$ denote the (ordinary universal) enveloping algebra of L. The augmentation ideal of L is $\mathcal{A}U(L) = LU(L)$.

LEMMA 4.2: *Let L be a Lie algebra with ordered basis $\mathcal{B}$ chosen so that, for each n, $\gamma_n(L)$ is linearly spanned by the $b \in \mathcal{B}$ with $\nu(b) \geq n$. Then, for each positive integer t, the set of Poincaré–Birkhoff–Witt monomials*

$$\mathcal{B}_t = \{b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_s^{\alpha_s} | b_1 < \cdots < b_s, \alpha_i \geq 1, s \geq 1, \sum_{i=1}^{s} \alpha_i \nu(b_i) \geq t\}$$

forms a basis of $(\mathcal{A}U(L))^t$.

Proof: For each integer $t \geq 1$, let E_t denote the F-linear span of all the products of the form

$$x_1 x_2 \cdots x_m$$

for some $m \geq 1$, where $x_1, \ldots, x_m \in L$ and $\nu(x_1)+\cdots+\nu(x_m) \geq t$. By Proposition 3.1 in [R2], $\mathcal{B}_t$ is a basis for E_t. It remains only to verify $E_t = (\mathcal{A}U(L))^t$. Clearly $E_1 = \mathcal{A}U(L)$. Because $E_i E_j \subseteq E_{i+j}$ for all i, j, it follows that $(\mathcal{A}U(L))^t = E_1^t \subseteq E_t$. To see the reverse inclusion, let $x = b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_s^{\alpha_s}$ be a typical element in the basis $\mathcal{B}_t$ of E_t. Then $b_i \in \gamma_{\nu(b_i)}(L) \subseteq (\mathcal{A}U(L))^{\nu(b_i)}$ for each i. Hence,

$$x \in (\mathcal{A}U(L))^{\sum_{i=1}^{s} \alpha_i \nu(b_i)} \subseteq (\mathcal{A}U(L))^t,$$

as required. ∎

LEMMA 4.3: *Let A be an associative algebra generated by a finite subset X, and let L be the Lie subalgebra of A generated by X. Then the following implications hold:*

1. *If L is finite dimensional and every commutator in X is algebraic in A, then A is finite dimensional.*
2. *If L is nilpotent and every commutator in X is nilpotent in A, then A is nilpotent.*

Proof: (1) Because L is assumed to be finite dimensional, we may choose finitely many commutators $\rho_1, \ldots, \rho_m$ in X such that $\mathcal{B} = \{\rho_1, \ldots, \rho_m\}$ is an ordered basis of L satisfying the hypotheses of Lemma 4.2. Consequently, for each $t \geq 1$, we may construct the basis $\mathcal{B}_t$ of $(\mathcal{A}U(L))^t$ described therein.

Since L generates A as an associative algebra, it follows from the universal property of enveloping algebras that there exists an associative algebra epimorphism $\mathcal{A}U(L) \to A$. The image of $\mathcal{B}_1$, therefore, linearly spans A. But, by assumption, each ρ_i is algebraic of degree d_i, say. Certainly then A is less than $(d_1 d_2 \cdots d_m)$-dimensional, proving (1).

(2) Let c denote the nilpotence class of L. Because L is finitely generated and nilpotent, it is certainly finite dimensional. This time we may choose the basis $\mathcal{B} = \{\rho_1, \ldots, \rho_m\}$ of commutators to satisfy $\rho_i^{d_i} = 0$ for each i. Notice, too, that $\nu(\rho_i) \leq c$ for each i. These facts force the image in A of each monomial in $\mathcal{B}_{c(d_1+\cdots+d_m-m)+1}$ to be trivial. Hence $A^{c(d_1+\cdots+d_m-m)+1} = 0$. ∎

As already mentioned the main ingredients in our proof are deep results due to Zelmanov. He proved in [Z1] that any Lie algebra over a field of characteristic zero with algebraic adjoint representation is locally finite-dimensional provided it satisfies a polynomial identity. In fact, according to Zelmanov, his proof required only that the commutators in the generating set be ad-algebraic:

THEOREM 4.4 ([Z1]): *Let L be a Lie algebra over a field of characteristic zero generated by a finite set X. Suppose that*

1. *every commutator in X is ad-algebraic, and*
2. *L satisfies a polynomial identity.*

Then L is finite-dimensional.

In the positive characteristic case, Zelmanov later proved:

THEOREM 4.5 ([Z2; Theorem 1.7]): *Let L be a Lie algebra over a field of positive characteristic generated by a finite set X. Suppose that*

1. *every commutator in X is ad-nilpotent, and*
2. *L satisfies a polynomial identity.*

Then L is nilpotent.

The characteristic zero analogue of Theorem 4.5 can be deduced from Theorem 4.4. Indeed, let L be a Lie PI-algebra over a field of characteristic zero generated by a finite set X such that every commutator in X is ad-nilpotent. Then L is finite-dimensional by Theorem 4.4. Thus, $\mathrm{ad}(L)$ is a finite-dimensional Lie algebra generated by the finite set S of all ad-maps of the form $\mathrm{ad}\,\rho$, where ρ is a commutator in X. Clearly, S is a nil subset of $\mathrm{End}_F(L)$ that is closed under Lie multiplication. Hence, by Jacobson's form of Engel's theorem (see [SF; Theorem 1.3.1], for example), $\mathrm{ad}(L)$ is nilpotent. This forces L to be nilpotent, too.

The following corollary completes our proof of Theorem A.

COROLLARY 4.6: *Let A be an associative algebra over a field F of characteristic $p \geq 0$ generated by a finite subset X, and let L be the Lie subalgebra of A generated by X. Then the following implications hold:*

1. *If $p = 0$ and every commutator in X is algebraic in A, then A is finite-dimensional provided L satisfies some polynomial identity.*

2. *If $p \geq 0$ and every commutator in X is nilpotent in A, then A is nilpotent provided L satisfies some polynomial identity.*

Proof: Assume that $p = 0$, every commutator in X is algebraic in A, and L is a PI-algebra. Then, by Lemma 4.1, every commutator in X is ad-algebraic. Theorem 4.4 now applies yielding the finite-dimensionality of L. But then A is finite-dimensional by Lemma 4.3. This proves part (1); the proof of part (2) is similar. ∎

In order to deduce Theorem B from Theorem A, we shall require the following lemma.

LEMMA 4.7: *Let A be a radical algebra that is finitely generated as a left A-module. If $\mathrm{gr}(A)$ is nilpotent then A is nilpotent.*

Proof: Let $a_1, \ldots, a_d$ generate A as an A-module. Then $A = Ba_1 + \cdots + Ba_d$, where B is the unital hull of A. We claim

$$A^k = \sum_{n_1,\ldots,n_k \in \{1,\ldots,d\}} Ba_{n_1}a_{n_2}\cdots a_{n_k}.$$

Indeed, by induction on k we have

$$\begin{aligned} A^{k+1} &= A \sum_{n_1,\ldots,n_k \in \{1,\ldots,d\}} Ba_{n_1}a_{n_2}\cdots a_{n_k} \\ &= \sum_{n_1,\ldots,n_k \in \{1,\ldots,d\}} BAa_{n_1}a_{n_2}\cdots a_{n_k} \\ &= \sum_{n_1,\ldots,n_k \in \{1,\ldots,d\}} B(Ba_1 + \cdots + Ba_d)a_{n_1}a_{n_2}\cdots a_{n_k} \\ &= \sum_{n_1,\ldots,n_{k+1} \in \{1,\ldots,d\}} B^2 a_{n_1}a_{n_2}\cdots a_{n_k}a_{n_{k+1}} \\ &= \sum_{n_1,\ldots,n_{k+1} \in \{1,\ldots,d\}} Ba_{n_1}a_{n_2}\cdots a_{n_{k+1}}. \end{aligned}$$

So, in particular, each A^k is d^k-generated as a B-module.

Clearly $\mathrm{gr}(A)$ is nilpotent if and only if $A^i = A^{i+1}$ for some i. Put $N = A^i$. Then, as shown above, N is a finitely generated B-module such that $N = AN = (\mathcal{J}B)N$. By Nakayama's Lemma this is impossible unless $N = 0$. ∎

Now let A satisfy the hypotheses given in Theorem B. Since A is finitely generated as an A-module, A_1 is finite-dimensional, so that $\mathrm{gr}(A)$ is finitely generated. Applying Theorem A yields that $\mathrm{gr}(A)$ is nilpotent. Finally, Lemma 4.7 completes the proof of Theorem B.

5. Additional remarks

We have defined the concept of being infinitesimally PI for groups, associative algebras and restricted Lie algebras. As mentioned at the end of Section 2, this property is not generally inherited by subobjects; however, we do have the following:

LEMMA 5.1: *The property of being infinitesimally PI is inherited by homomorphic images.*

Proof: Consider the case of an epimorphism $A \to B$ of associative algebras. This map induces an associative algebra epimorphism $\mathrm{gr}(A) \to \mathrm{gr}(B)$, which when restricted to the generating component yields a Lie algebra epimorphism $\mathcal{L}(A) \to \mathcal{L}(B)$. The result follows in this case.

Next consider an epimorphism of groups $G \to G/N$. This map induces an epimorphism of associative algebras $F[G] \to F[G/N] \cong F[G]/(\mathcal{A}F[N])F[G]$. It follows from the preceding paragraph that this induces a Lie algebra epimorphism:

$$F \otimes_{\mathbb{F}_p} \mathrm{gr}(G) \cong \mathcal{L}(\mathcal{A}F[G]) \to \mathcal{L}(\mathcal{A}F[G/N]) \cong F \otimes_{\mathbb{F}_p} \mathrm{gr}(G/N).$$

Thus, if $\mathrm{gr}(G)$ satisfies some multilinear identity, then so does $\mathrm{gr}(G/N)$.

The restricted Lie algebra case is similar. ∎

Let A be a radical algebra generated by a subset X over a field F of positive characteristic. Below, we indicate by G the subgroup generated by X in the adjoint group $(A, \circ)$ of A given by $a \circ b = a + b + ab$. We write L for the restricted Lie subalgebra of A generated by X.

LEMMA 5.2: *Under the conventions described above:*

1. *If G is infinitesimally PI then A is infinitesimally PI.*
2. *If L is infinitesimally PI then A is infinitesimally PI.*

Proof: We prove only (1), the proof of (2) being similar. Because G generates the unital hull of A as an associative algebra, there is an epimorphism of associative algebras $\mathcal{A}F[G] \to A$. Recall G is infinitesimally PI if and only if $\mathcal{A}F[G]$ is infinitesimally PI. Now apply Lemma 5.1. ∎

Lemma 5.2 and the Corollary to Theorem B combine to prove our last result:

COROLLARY 5.3: *If A is a nil algebra generated by a finite subset X, then any one of the following conditions implies that A is nilpotent:*

1. $(A, \circ)$ *satisfies a law.*
2. G *satisfies a law.*
3. G *is infinitesimally PI.*
4. L *is a PI-algebra.*
5. L *is infinitesimally PI.*
6. A *is a PI-algebra.*
7. A *is infinitesimally PI.*

It follows that the infinite finitely generated p-groups of Golod type are not infinitesimally PI.

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